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THE GLUEBALL;
THE FUNDAMENTAL PARTICLE OF NON-PERTURBATIVE QCD

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ABSTRACT

Theoretical ideas related to the existence of glueballs in QCD are reviewed. These include non-perturbative phenomena such as confinement, instantons, vacuum condensates and renormalons. We also discuss glueball dominance of the trace of the stress-tensor, the mass content of the nucleon and a theorem on the lightest glueball state.

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Glueballs are perhaps the most dramatic and novel prediction of QCD. From the vantage point of twenty years ago when QCD was first being proposed as the fundamental theory of the strong interactions, the idea that there might be quarkless hadronic states whose constituents were massless gauge bosons (i.e. gluons) was almost revolutionary. Glueballs are inherently quantum chromodynamic in nature and, as such, their existence is closely related to other essentially non-perturbative phenomena that dominate low-energy hadronic physics such as the existence of vacuum condensates and the dominance of glue in determining the gravitational mass of visible matter. They clearly play a central role in elucidating QCD and their discovery would certainly be of great significance. Indeed had such particles been found 15-20 years ago, their dicoverers would certainly have been prime candidates for a Nobel Prize. Unfortunately, however, no unambiguous experimental signal for their existence has thus far been found. This is due in large part to the fact they can readily mix with ordinary quark model states and so can only be identified by a process of elimination, i.e. by searching for extra states beyond conventional "naive" quark model ones which have the correct decay characteristics. There has recently been a renewed flurry of interest, both experimental and theoretical, in these very interesting states and the situation is, in fact, beginning to clarify [1]–[7]. Much detailed analysis has been performed on a large amount of experimental data with the result that a few rather good candidates have emerged particularly in the region 1.5-1.7GeV [1][2][3]. In spite of this, however, the situation still remains unresolved and and more work needs to be done.

The theoretical situation is similarly somewhat ambiguous. Potential, bag and instanton gas models do indeed indicate that the lowest state should be a scalar (and not a pseudoscalar or tensor, for example) and that its mass should be in the above range [4][6][7][8]. All of these models, in spite of having the virtue of incorporating the correct low energy physics of QCD, are only effective representations of the full theory, and so their accuracy is difficult to evaluate. Recent intensive lattice simulations of QCD focussed explicitly on the glueball are in general agreement with the results of these models [5]. On the other hand, estimates from QCD sum rules indicate that the pseudoscalar rather than the scalar should be the lowest state albeit with a mass also in the general range of 1.5GeV [9]. In addition there are field theoretic models in which the 2^{++} tensor is the lightest state[10]. This disagreement between the QCD sum rules and the lattice estimates is surprising since these ought to be the least model dependent and therefore the most reliable. However, the lattice simulations do use a quenched, or valence, approximation, though it is generally believed that this is not a major source of error, and the QCD sum rules have difficulty

satisfying a low energy theorem. Below I shall prove a theorem that shows that, regardless of the model or approximation used, QCD requires that the scalar must, in fact, be the lightest glueball state. As a corollary various mass inequalities such as $M(2^{++}) \geq M(2^{-+})$ can also be proven.

Most of this paper will be devoted to a general overview of some of the theoretical ideas that impact the glueball question and its relationship to QCD. I shall try to emphasise some issues and developments that have not received quite as much attention in this context as some of the more well-known topics such as quark and bag models, lattice gauge theory and so on. Among the topics that I shall address are the operator description of the states, low energy theorems, glueball dominance of the stress-energy tensor and its relationship to the gluon dominance of the proton mass. The self-interaction of the gluons reflects the non-abelian gauge character of QCD; this is the origin of both the possibility that there are glueball states as well as of the phenomenon of asymptotic freedom. The latter is a property of the perturbative sector of the theory whereas the former is a product of the non-perturbative. Furthermore, both of these remarkable phenomena arise in the purely gauge sector of QCD and do not require the existence of quark degrees of freedom. Since glueballs are inherently non-perturbative in nature their existence is closely related to color confinement and to the existence of vacuum condensates and instantons. It is in this sense that they can be dubbed the "fundamental particles" of non-perturbative QCD. Ultimately one would like to be able to start with the QCD Lagrangian and derive its spectrum in some well-defined approximation scheme. Thus far this has proven impossible in spite of ambitious attempts such as the large N_c expansion, chiral perturbation theory, soliton models, heavy quark expansions, instanton gas models and so on. Apart from some recent work on the latter [4] these methods focus on the quark sector and have had little to say about the glueball spectrum. Only lattice gauge theory [5] and, to some extent, the sum rule consistency relations [9] can be said to have provided some direct contact with fundamental QCD. Otherwise most of our intuition and predictions about glueballs are derived from models.

Within the field theoretic framework of QCD all hadronic states are created by composite operators constructed out of fundamental quark and gluon fields. Some well-known

examples are the following:

$$\begin{array}{rl} \text{Scalars} & \sigma_a(x) \propto \bar{q}(x)\lambda_a q(x) \\ \text{Pseudoscalars} & \phi_a(x) \propto \bar{q}(x)\gamma_5\lambda_a q(x) \\ \text{Vectors} & \rho_a^\mu(x) \propto \bar{q}(x)\gamma^\mu\lambda_a q(x) \\ \text{Glueball} & G(x) \propto F_{\mu\nu}^a(x)F_a^{\mu\nu}(x) \\ \text{Glueball} & \tilde{G}(x) \propto F_{\mu\nu}^a(x)\tilde{F}_a^{\mu\nu}(x) \end{array}$$

By analogy with the ordering of operators in the operator product expansion it is natural to order these by dimension. It was originally suggested by both Bjorken and Jaffe et al. [11] that, at least heuristically, one might expect the mass of a state to increase with the dimension of the corresponding lowest dimensional operator that can produce it. In the table below an obvious shorthand is used to describe the operators: Γ represents a gamma matrix, D the covariant derivative and F the gluon field tensor. The most salient feature of this is that all of the conventional quark model states are indeed those of lowest dimension while the exotic states, namely the glueball, hybrid and "molecular-like" ones are of higher dimension. Though suggestive this does not explain why the quark model states should so dominate the low energy spectrum. Notice also that there are many states with the same quantum numbers arising from quite different operators leading to the complication of untangling the "pure" states from the physical states. On the other hand the lowest hybrid operator does give rise to a state which cannot occur in the quark model, the 1^{-+}. An unambiguous discovery of such a state in the low energy spectrum would indeed have been a major triumph for QCD.

Dimension	Operator	J^{PC}	Character
3	$\bar{q}\Gamma q$	$0^{-+}, 1^{--}, 0^{++}, 1^{+-}, 1^{++}$	QuarkModel
4	$\bar{q}\Gamma Dq$	$2^{++}, 2^{-+}, 2^{--}$	QuarkModel
4	F^2	$0^{++}, 0^{-+}, 2^{++}, 2^{-+}$	Glueball
5	$\bar{q}\Gamma Fq$	$0^{-+}, 1^{-+}, 0^{++}, 2^{-+}$	Hybrid
6	F^3	$0^{++}, 0^{-+}, 1^{+-}, 3^{+-}$	Glueball
6	$\bar{q}\Gamma q\bar{q}\Gamma q$	0	"Molecules"

To understand somewhat more quantitatively why glueballs, for example, should have a higher mass than a typical light quark state it is useful to use the language of a potential

or bag model. The argument I shall present is very simple and shouldn't be taken too seriously though it is useful for giving some insight into what the important physics is at work here [12]. There are many variants of the color-force potential but all of them have two major characteristics in common corresponding roughly to the perturbative and non-perturbative aspects of the theory: a Coulomb-like piece and a long-range confining piece. A simple qualitative representation is

$$V(r) = -\frac{\alpha}{r} + \sigma r \tag{1}$$

where $\alpha \approx 0.2$ and σ (the string tension) $\approx 400MeV$. In QCD there is, of course, only a single scale parameter, namely the running coupling constant $\alpha_s(\mu)$ defined at some scale μ. All of the parameters of an effective potential, such as α and σ occurring in eq. (1), are, in principle, expressible in terms of $\alpha_s(\mu)$. In the simplest version of the quark model this potential is used in a Schrodinger equation with quarks whose effective mass is roughly 300MeV. One of the great mysteries of QCD is that this prescription gives a remarkably good accounting of the low-lying hadrons. In QED (the limit $\sigma = 0$, $\alpha = e^2$ in eq. (1)) the total energy is given by

$$E = \frac{p^2}{2m} - \frac{e^2}{r} \tag{2}$$

where p is the momentum and m the mass. From the uncertainty principle $pr \geq 1$, so

$$E \geq \frac{1}{2mr^2} - \frac{e^2}{r} \tag{3}$$

Minimising this lower bound gives $E_{min} = -me^2/2$ with $r_{min} = 1/me^2$ which agree with the ground state values for the hydrogen atom. Let us apply this to the glueball considered as a bound state of two massless gluons:

$$E = 2p + \frac{9}{4}\sigma r - \frac{\alpha}{r} \tag{4}$$

The factor 9/4 is simply a color factor. Minimising as before leads to $r = 2/3[(2-\alpha)/\sigma]^{1/2}$ and $E = 3[(2-\alpha)\sigma]^{1/2} \approx 3\sqrt{2}\sigma^{1/2} \approx 1.7$GeV. Not surprisingly this shows that the glueball mass is governed by the non-perturbative string tension. Furthermore, even though $\sqrt{\sigma} \approx$ 400MeV sets the scale, it also shows that the expected mass of the lightest glueball is quite large, between 1.5 and 2GeV. A similar calculation can be performed for a typical meson. The analog to eq. (4) is

$$E = 2(p^2 + m^2)^{1/2} - 2m + \sigma r - \frac{\alpha}{r} \tag{5}$$

which leads to $E_{min} \approx 750$MeV. It is also possible to extend this argument to hybrids by considering them as bound states of two massive quarks and a massless glueball; similar calculations to the above indicate that their lowest mass is in the range of 2.5GeV. This argument therefore shows that glueballs should be heavier than light quark states but lighter than hybrids.

The discretized version of these composite operators (or a smeared out version of them) is what is used in lattice gauge theory to simulate the behaviour of the corresponding propagators thereby allowing a "measurement" of the relevant hadronic mass. As already remarked there has been a significant amount of work done using this approach to study the glueball. The most intensive study [5]reveals that the 0^{++} should have a mass of approximately 1.7GeV somewhat higher than those considered to be the best experimental candidates (at approximately 1.5GeV) [1][2][3]; however, these are within experimental (and presumably theoretical!) limits.

Before discussing QCD sum rules, instantons and the like it is worth digressing here to emphasise the special role played by the glueball in QCD beyond that of the "hydrogen atom of non-perturbative physics". Recall first that the glueball field

$$G(x) = f_G F^a_{\mu\nu}(x) F^{\mu\nu}_a(x) \tag{6}$$

is identical, up to constant factors, to the Lagrangian density of the pure gauge sector. Furthermore, it is also identical to the trace of the stress-energy tensor, θ, which is the operator that determines masses of particles. The renormalisation of the trace anomaly in the triangle graph occurring in the θgg vertex leads to

$$\theta = \sum m_q \bar{q} q + \frac{\beta(g)}{g} F^2_{\mu\nu} \tag{7}$$

where $\beta(g)$ is the conventional β function: $\beta(g) = -bg^2 + \ldots$ with $b = (11 - 2n_f)/48\pi^2$. Thus, even in massless QCD hadrons can be massive since $\theta \neq 0$. Indeed, eq. (7) naturally leads to the idea of "glueball dominance of the trace of the stress tensor" (at least when quark masses can be neglected):

$$\theta(x) = f_G m_G^2 F^2_{\mu\nu}(x) = m_G^2 G(x) \tag{8}$$

Notice that $f_G m_G^2 = \beta(g)/g$. Eq. (8) is the exact analog of both PCAC ($\partial_\mu A_\mu = f_\pi m_\pi^2 \phi_\pi$) and vector dominance of the electromagnetic current ($J_\mu = f_\rho m_\rho^2 \rho_\mu$). By taking matrix elements of (8) between hadronic (H) states at rest and using the fact that

$$\langle p|\theta|p\rangle = M(Baryons); \quad 2m^2(Mesons) \tag{9}$$

Goldberger-Treiman type relations can be derived [13]. Generically, these are of the form $f_G g_{GHH} \approx M_H$. The continuation in mass to the physical region is quite severe here; however, this does allow a rough estimate of hadronic couplings relevant to experimental searches. Since the stress tensor itself generates the full Poincaré algebra and, in particular, $\theta = \partial_\mu D_\mu$, where $D_\mu = x^\nu \theta_{\mu\nu}$ is the dilation current which is the generator of scale transformations, the glueball is part of a rich algebra (akin to chirality) from which low energy theorems can be derived. For example, one such theorem is $f_G^2 m_G^2 \approx 16\pi b \alpha_s E^4$ where

$$E \equiv \langle 0|G(0)|0\rangle \tag{10}$$

is the energy density of the glueball vacuum condensate.

Another interesting example is provided by the mass of the nucleon: since the masses of the u and d quarks are only a few MeV and heavy quarks are not a major component of the nucleon almost all of its mass must be derivable from the gluon field. Put slightly differently: if there were no gluon component in eq. (7) the nucleon would weigh only a few MeV! Thus $M_N \approx (\beta(g))/g\langle p|F_{\mu\nu}^2|p\rangle$. This, in fact, is not quite right because heavy quarks can, in fact, contribute to (7) through a triangle graph which then connects to the nucleon through gluons; (this is effectively the sea contribution)[14]. In the limit $m_q \to \infty$ this gives

$$\langle p|\sum m_q \bar{q}q|p\rangle \approx -\frac{n_h g^2}{24\pi^2}\langle p|F_{\mu\nu}^2|p\rangle \tag{11}$$

where n_h is the number of heavy quark flavours. This contribution exactly cancels the heavy quark contribution in the β function so $M_N \approx (\beta_l(g)/g)\langle p|F_{\mu\nu}^2|p\rangle$ where the subscript l indicates that only light flavours are to be counted in β. This is an elegant example of the decoupling theorem at work. Because of eqs. (6) and (8) this formula explicitly exhibits glueball dominance in determining masses of light hadrons.

The role of the s-quark is ambiguous in this analysis since its mass is comparable to the perturbative scale. Its contribution, $\langle p|m_s \bar{s}s|p\rangle$, can be estimated from the sum rule for the nucleon sigma term and the Gell-Mann-Okubo formula for symmetry breaking of baryon masses. The upshot of a careful analysis is that it contributes about 30% of the mass, most of the rest being from glue and only a few per cent actually being derived from the light u and d quarks[15]. This situation is reminiscent of the ambiguities in interpretation of the origin of the nucleon spin and, indeed, both the s-quark and a triangle anomaly play important roles in both analyses. The “paradoxical” nature of these problems can be highlighted by observing that, if one neglects the s-quark contribution, then the nucleon

mass can be expressed as $M_N = [(33-2n_l)/2n_h]\langle p|\sum m_h\bar{h}h|p\rangle$ which would seemingly imply that it is derived solely from its heavy quark content! Of course the decoupling theorem obtained through the triangle graph shows that this is, in fact, identical to the purely (low-energy) gluon contribution as in eq. (11). Care must therefore be taken in how these formulae are interpreted.

The mass of the glueball is determined by the leading singularity in its propagator which, if the glueball is stable, is just a simple pole. Both the mass and the propagator satisfy renormalisation group (RG) equations. Consider massless QCD, then the only scale in the problem is the renormalisation scale, μ, needed to specify the physical coupling, $g(\mu)$, so, on dimensional grounds

$$m_G = \mu f[g(\mu)] \tag{12}$$

Since μ is arbitrary, $dm_G/d\mu = 0$ which leads to the most elementary RG equation

$$\frac{d\ln f}{dg} = \frac{1}{\beta(g)} \tag{13}$$

and, therefore,

$$m_G = c_G\mu exp\int\frac{dg}{\beta(g)} \equiv c_G\Lambda_{QCD} \approx c_G\mu e^{1/bg^2} \tag{14}$$

where c_G is a constant that determines the glueball mass in terms of Λ_{QCD} . In the second part of this equation the perturbative expansion for $\beta(g)$ has been used. Eq. (14) shows explicitly how mass can be generated in a massless theory ("dimensional transmutation") and, more significantly here, that it is is inherently non-perturbative. Notice, however, that this non-perturbative behaviour is generated from perturbative effects via renormalisation and characteristically leads to e^{1/bg^2}. This behaviour is called the renormalon contribution by analogy with that of the instanton which has a characteristic $e^{8\pi^2/g^2}$ behaviour. Instantons arise from non-trivial local minima of the action. For example, consider the scalar correlator

$$\Gamma(\mathbf{x},t) \equiv \langle 0|T[G(\mathbf{x},t)G(0)]|0\rangle \tag{15}$$

which has a standard path integral representation [16]:

$$\Gamma(\mathbf{x},t) = \int\mathcal{D}A^a_\mu e^{\frac{i}{4}\int F^a_{\mu\nu}F^{\mu\nu}_a d^4x} det(\not{D}+m)G(\mathbf{x},t)G(0) \tag{16}$$

An expansion of its Fourier transform, $\Pi(q^2/\mu^2, g^2)$, in terms of g^2 is generically of the form:

$$\Pi\left(\frac{q^2}{\mu^2}, g^2\right) = \sum_{n=0}^{\infty} a_n(q^2)g^{2n} + \sum_{m,n=0}^{\infty} e^{-8\pi^2(m+1)/g^2} b_{mn}(q^2)g^{2n} \tag{17}$$

The first term represents ordinary perturbation theory (i.e. an expansion around the trivial vacuum where the action vanishes) and the second an expansion around instantons whose action is an integral multiple of $8\pi^2$.

A Kallen-Lehmann representation for $\Gamma(\mathbf{x}, t)$ can be inferred from asymptotic freedom and the fact that $G(x)$ is of dimension 4:

$$\Gamma(\mathbf{x}, t) = \Pi'(0, g^2)\partial^2\delta^{(4)}(x) + \Pi(0, g^2)\delta^{(4)}(x) + \partial^4 \int_{M_G^2}^{\infty} \frac{dq^2}{q^4}\rho(q^2/g^2, g^2)\Delta_F(x, q^2) \quad (18)$$

Here $\rho(q^2)$ is the spectral weight function and $\Delta_F(x, q^2)$ the standard free Feynman propagator. Correspondingly,

$$\Pi\left(\frac{q^2}{\mu^2}, g^2\right) = \Pi(0, g^2) + q^2\Pi'(0, g^2) + q^4 \int_{M_G^2}^{\infty} \frac{dq'^2\rho(q'^2/\mu^2, g^2)}{q'^4(q'^2 - q^2)} \quad (19)$$

This dispersion relation and its implied high energy perturbative contribution is the starting point for the QCD sum rule consistency conditions. The right-hand-side is saturated with known, or presumed, resonances (the various glueball and quark mesonic states) and its high energy tail by a perturbative contribution derived from asymptotic freedom. On the left-hand -side the operator product expansion is used to express $T[G(\mathbf{x}, t)G(0)]$ in terms of a complete set of operators of increasing dimension. In pure QCD this gives rise to a series with the (symbolic) structure:

$$\Pi\left(\frac{q^2}{\mu^2}, g^2\right) = b_1\langle 0|F_{\mu\nu}^2|0\rangle + b_2\langle 0|F_{\mu\nu}^3|0\rangle + b_3\langle 0|F_{\mu\nu}^4|0\rangle + \ldots \quad (20)$$

where the coefficients b_n are calculable. Masses of hadronic states are then related to the vacuum condensates occuring in this equation; (the first of these is essentially E of eq. (10)). For the glueball channel a detailed analysis has been carried out by Narison and Veneziano [9] who concluded that the ground state is the 0^{+-} rather than the 0^{++} expected from naive potential and bag models as well as from an intense lattice gauge simulation. As already remarked we shall prove below that, at least in pure QCD, the 0^{++} must be the lightest state. Before doing so it is worth remarking that the general constraints imposed on the propagator (and, therefore, implicitly the mass) by the RG, analyticity and the existence of a perturbative regime are non-trivial to satisfy [17]. Roughly speaking, the RG forces $\Pi(q^2/\mu^2, g^2)$ to be a function of the single variable $(q^2/\mu^2)exp \int \frac{dg}{\beta(g)}$, rather than

of the two variables q^2 and g^2 separately, as in a perturbative Feynman graph expansion. Thus, if it is analytic in q^2 and there is a mass gap, it cannot be analytic in g^2 so the perturbative expansion must diverge and be, at best, asymptotic. This suggests that there must be some subtle interplay between the perturbative and non- perturbative, somehow "mediated" by the renormalon contribution. One might, therefore, be able to improve the sum rule predictions by enforcing the RG constraint; effectively, this amount to including renormalon contributions.

Let us now show that the lightest glueball must be the 0^{++}. Consider the quantity (for $t > 0$)

$$Q(t) \equiv \int d^3x \Gamma(\mathbf{x}, t) \tag{21}$$

$$= \sum_N |\langle 0|G(0)|N\rangle|^2 \delta^{(3)}(\mathbf{p_N}) e^{iM_N t} \tag{22}$$

where M_N is the invariant mass of the state $|N\rangle$. The Euclidean version of this (effectively given by taking $t \rightarrow i\tau$) implies that, when $\tau \rightarrow \infty$,

$$Q_E(\tau) \equiv Q(i\tau) \approx e^{-M_0 \tau} \tag{23}$$

where M_0 is the mass of the lightest contributing state. An analogous result can be derived from the Euclidean version of eq. (18) for the asymptotic behaviour of the full correlator when either τ or $|x|$ become large. Up to powers, this simply reflects the exponential decay of $\Delta_F(x, \mu^2)$ in the deep Euclidean region. This asymptotic behaviour in Euclidean space forms the basis for extracting particle masses from lattice QCD simulations [5] and will similarly play a central role in our proof. There are a couple of points worth remarking about it before proceeding. First, in pure QCD, where the scalar and pseudoscalar glueballs are expected to be the lightest states in their respective channels, $M_0 = M_G$ or $M_{\tilde{G}}$. In the full theory, however, the corresponding lightest states are those of 2 pions and 3 pions, respectively, and even the lightest glueballs become unstable resonances. In that case $M_0 = M_{2\pi}$ or $M_{3\pi}$. On the other hand, in the limit when τ becomes large, but remains smaller than $\sim 2M_G/\Gamma_G^2$, where Γ_G is the width of the resonance, one can show that the exponential decay law, eq. (23), still remains valid but with a mass M_0 given by M_G rather than $M_{2\pi}$; (a similar result obviously also holds for the pseudoscalar case). The point is that, if there are well-defined resonant states present in a particular channel, then they can be sampled by sweeping through an appropiate range of asymptotic τ values where they dominate, since τ is conjugate to M_N [18].

The basic inequality that we shall employ is that, in the Euclidean region,

$$(F^a_{\mu\nu} \pm \tilde{F}^{\mu\nu}_a)^2 \geq 0 \qquad \Rightarrow \qquad f_G^{-1} G_E(\mathbf{x},\tau) \geq \pm f_{\tilde{G}}^{-1} \tilde{G}_E(\mathbf{x},\tau) \tag{24}$$

where $G_E(\mathbf{x},\tau) \equiv G_E(\mathbf{x},it)$. The integral version of this will be recognised as the original basis for proving the existence of instantons, to which we shall return below. Although this inequality holds for classical fields, it can be exploited in the quantized theory by using the path integral representation, eq. (16), in Euclidean space where the measure is positive definite. The positivity of the measure has been skillfully used by Weingarten [19] to prove that in the quark sector the pion must be the lightest state. Here, when combined with the inequality (24), it immediately leads to the inequalities (valid for $\tau > 0$)

$$f_G^{-2} \Gamma_E(\mathbf{x},\tau) \geq f_{\tilde{G}}^{-2} \tilde{\Gamma}_E(\mathbf{x},\tau) \qquad \text{and} \qquad f_G^{-2} Q_E(\tau) \geq f_{\tilde{G}}^{-2} \tilde{Q}_E(\tau) \tag{25}$$

By taking τ large (but $< 2M_G/\Gamma_G^2$) and using (23), the inequality

$$M_G \leq M_{\tilde{G}} \tag{26}$$

easily follows. In pure QCD where these glueballs are isolated singularities, their widths vanish and the limit $\tau \to \infty$ can be taken without constraint.

Although this is the result we want, its proof ignored the existence of the vacuum condensate E, eq. (10). Since $E \neq 0$ the vacuum is the lightest state contributing to the unitarity sum so $M_0 = 0$ and the large τ behaviour of $\Gamma_E(\mathbf{x},\tau)$ is a constant, E^2, rather than an exponential. Thus, the inequalities (25) are trivially satisfied for asymptotic values of τ since there is no condensate in the pseudoscalar channel. It is, incidentally, the occurrence of M_G in a sub-leading asymptotic role masked by this constant condensate term that makes its extraction from lattice data so challenging. To circumvent this problem it is clearly prudent to consider either the derivative of $Q(t)$ or, more generally, the time or space evolution of $\Gamma(\mathbf{x},t)$ since these remove the offending condensate contribution. Although many of the subtleties can be finessed by considering $\nabla^2 \Gamma_E(\mathbf{x},\tau)$ it is instructive to first consider (for $\tau > 0$)

$$\dot{Q}_E(\tau) = -\sum_N |\langle 0|G(0)|N\rangle|^2 \delta^{(3)}(\mathbf{p_N}) M_N e^{-M_N \tau} \tag{27}$$

The vacuum state clearly does not contribute to this so its large τ behaviour is, up to a factor $-M_0$, just that of eq. (23) except that M_0 is now the mass of the lightest contributing particle state. Now, (for $\tau > 0$),

$$\Gamma_E(\mathbf{x},\tau) = \langle 0|e^{H\tau} G_E(0) e^{-H\tau} G_E(0)|0\rangle \tag{28}$$

which implies

$$\dot{\Gamma}_E(\mathbf{x},\tau) = -\langle 0|G_E(\mathbf{x},\tau)HG_E(0)|0\rangle \tag{29}$$

where, in the last step, the condition $H|0\rangle = 0$ has been imposed. Notice that, whereas both $Q_E(\tau)$ and $\Gamma_E(\mathbf{x},\tau)$ are positive definite, their time derivatives are negative definite. Now, at the classical level H is positive definite. We can therefore repeat our previous argument by working in Euclidean space and combining the inequalities (24) with a path integral representation for (29) to formally obtain (for $\tau > 0$) the inequalities

$$f_G^{-2}\dot{\Gamma}_E(\mathbf{x},\tau) \le f_{\tilde{G}}^{-2}\dot{\tilde{\Gamma}}_E(\mathbf{x},\tau) \qquad \text{and} \qquad f_G^{-2}\dot{Q}_E(\tau) \le f_{\tilde{G}}^{-2}\dot{\tilde{Q}}_E(\tau) \tag{30}$$

The large τ limit then leads to

$$f_G^{-2}M_G e^{-M_G\tau} \ge f_{\tilde{G}}^{-2}M_{\tilde{G}} e^{-M_{\tilde{G}}\tau} \tag{31}$$

from which (26) follows even in the presence of condensates.

There are some subtle points in this argument that require clarification, in particular the nature of the path integral representation for (29) and the question of the vacuum energy contribution. These are best dealt with using the language and results of the transfer matrix formalism used in lattice theory since this is directly formulated in the Euclidean region as a Lagrangian theory where the measure is positive definite. Rather than showing how this can be done here, we shall instead circumvent these technical problems by considering the space rather than time evolution of Γ. To this end consider

$$\nabla^2\Gamma_E(\mathbf{x},\tau) = -\langle 0|G(\mathbf{x},\tau)\mathbf{P}^2G(0)|0\rangle \tag{32}$$

where $\mathbf{P} = \mathbf{E}_a \times \mathbf{B}_a$ is the 3-momentum operator. The asymptotic behaviour of the full correlator can be deduced from from its Kallen-Lehmann representation, eq. (19). From this one finds that the large τ behaviour of $\nabla^2\Gamma_E(\mathbf{x},\tau)$ is, up to powers, again $e^{-M_G\tau}$. The path integral representation of (32), in which $\mathbf{E}_a$ is replaced by $\dot{\mathbf{A}}_a$, is negative definite so all of the previous arguments go through leading to the inequality (26). Notice that, unlike the time derivative case, the vacuum energy presents no complication since $\mathbf{P}|0\rangle \equiv 0$.

The extension of the above argument to the general case showing that the scalar must be lighter than all other glueball states, can now be effected. Introduce an operator, $T_{\mu\nu\alpha\beta\ldots}(x)$, constructed out of a sufficiently long string of $F^a_{\mu\nu}(x)'s$ and $\tilde{F}^{\mu\nu}_a(x)'s$ that it can, in principle, create an arbitrary physical glueball state of a given spin. Generally

speaking a given T once constructed can, of course, create states of many different spins, depending on the details of exactly how it is constructed. As a simple example consider the fourth-rank tensor [20]

$$T_{\mu\nu\alpha\beta}(x) = F_{\mu\nu}(x)F_{\alpha\beta}(x) \tag{33}$$

which creates glueball states with quantum numbers 2^{++} and 0^{++}. Now, in Euclidean space, the magnitude of any component of $F^a_{\mu\nu}(x)$, or $\tilde{F}^{\mu\nu}_a(x)$, is bounded by the magnitude of $[F^a_{\mu\nu}(x)F^{\mu\nu}_a(x)]^{\frac{1}{2}}$. Hence, any single component of $T_{\mu\nu\alpha\beta}(x)$ must, up to a constant, be bounded by $G(x)$:

$$T_{\mu\nu\alpha\beta}(x) \leq f_G^{-1} G(x) \tag{34}$$

This inequality is the analog of (24) and so the same line of reasoning used to exploit that inequality when proving (26) can be used here. Following the same sequence of steps leads to the conclusion that M_G must be lighter than the lightest state interpolated by $T_{\mu\nu\alpha\beta}(x)$, from which the inequality

$$M(2^{++}) \geq M(0^{++}) \equiv M_G \tag{35}$$

follows. It is worth pointing out that the pseudoscalar analog of this operator can be similarly bounded thereby leading to the inequality $M(2^{++}) \leq M(2^{-+})$. This argument can be generalized to an arbitrary $T_{\mu\nu\alpha\beta\ldots}(x)$ since, again up to some overall constants analogous to f_G, it is bounded by some power (p) of $G(x)$; i.e., for any of its components, $T_{\mu\nu\alpha\beta\ldots}(x) \leq G(x)^p$. Now, the operator $G(x)^p$ has the same quantum numbers as $G(x)$ and so can also serve as an interpolating field for the creation of the scalar glueball. The same arguments used to prove that this 0^{++} state is lighter than either the 0^{+-} or the 2^{++} can now be extended to the general case showing that it must be lighter than *any* state created by *any* T; in other words, the scalar glueball must indeed be the lightest glueball state.

Finally, we make some brief remarks about the conditions under which the bound is saturated. Clearly the inequality (24) becomes an equality when

$$F^a_{\mu\nu}(x) = \tilde{F}^a_{\mu\nu}(x) \tag{36}$$

i.e. when $E^a_i(x) = B^a_i(x)$, which is also the condition that minimizes the action and signals the dominance of pure instantons. In such a circumstance the scalar and pseudoscalar will be degenerate. However, the proof of the mass inequality (26) only required (24) to be

valid at asymptotic values of $|x|$. Thus, the saturation of this bound actually only rests on the weaker condition that F be self-dual in the asymptotic region where it must vanish like a pure gauge field. Similarly, the saturation of the general inequality showing the scalar to be the lightest state occurs when *all* components of $F^a_{\mu\nu}(x)$ have the same functional dependence at asymptotic values of $|x|$. Although this is a stronger condition than required by the general asymptotic self-dual condition (36), it is, in fact, satisfied by the explicit single instanton solution that satisfies it. For instance, in $SU(2)$,

$$F_{\mu\nu}(x) = \frac{4\lambda^2}{x^2 + \lambda^2}\sigma_{\mu\nu} \tag{37}$$

Thus, the splitting of the levels is determined by how much the asymptotic behaviour of the non-perturbative fields differ from those of pure instantons. This therefore suggests a picture in which the overall scale of glueball masses is set by non-perturbative effects driven by instantons (thereby producing the confining long-range force) but that the level splittings are governed by perturbative phenomena.

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